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ON THE ACCUMULATION OF ERRORS AT NUMERICAL INTEGRATION IN SOME PROBLEMS OF CELESTIAL MECHANICS

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SUMMARY

The results are expounded of the application of Myachin's estimates of errors in numerical integration of the equation of motion in the case of a problem of **n** bodies. Several numerical examples are given.

author

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In connection with the rapid development of computing techniques the numerical integration methods have become the most effective when undertaking the solution of the problem of **n** bodies. Such colossal works by their volume of calculation as "Coordinates of the Five Outer Planets 1653 - 2060", " Coordinates of Four Minor Planets 1940 - 1960" and others, were completed by numerical integration methods. However, the merit of numerical methods is substantially diminished on account of error accumulation at integration.

The error occurs on account of limited precision in the computer (rounding-off errors), because of unaccounted differences in the integration formulas, and also on account of the inaccurate value of initial data.

The errors due to unaccounted differences may be reduced to a minimum by proper selection of interval and the number of terms in the integration formula. The accounting of errors in the initial data offers little difficulty, for if the initial data are computed with a precision, with which the computations are performed, they may be considered as rounding-off errors in the first integration step.

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*) O NAKOPLENII OSHIBOK PRI CHISLENNOM INTEGRIROVANII V NEKOTORYKH ZADACHAKH NEBESNOY MEKHANIKI.

In most of cases it is important to know the dependence between the number of steps and of vanishing columns, so that the required precision be assured beforehand, or the error with which either quantities are obtained well established.

With this in view V. F. Myachin (1959) derived formulas for the estimate of rounding off error accumulation when integrating numerically the equations of motion in the problem of two bodies.

If in these formulas we neglect the eccentricity e, and denote by $\delta_k^{(i)}$ the true error in the k-th step, we may write them in the form:

$$|\delta_k^{(i)}| < \varepsilon_k^{(i)} \qquad (k=1,\ 2,\ 3\ldots;\quad i=1,\ 2,\ 3),$$

where k is the number of the step, i is the number of the coordinate (x, y, z),

$$\varepsilon_k^{(i)} = \frac{\rho\sqrt{3}}{(nh)^{3/2}}\sqrt{N^{(i)}(E_k)},$$

$$\begin{aligned} N^{(i)} \equiv\ & 3\sigma_k^{(i)2}(E_k-E_0)^3 + 6\sigma_k^{(i)}\gamma_k^{(i)}(E_k-E_0)^2 + \Big[\frac{13}{2} - 6s_{i,3}^2 + \frac{15}{2}\sigma_k^{(i)2} + 12\sigma_k^{(i)2}\cos(E_k-E_0) + \\ & + 12\sigma_k^{(i)}\sigma_0^{(i)}\Big](E_k-E_0) + \Big[-8(1-s_{i,3}^2)\sin(E_k-E_0) - 24\sigma_k^{(i)2}\sin(E_k-E_0) + \\ & + 3(\sigma_k^{(i)}\gamma_k^{(i)} - \sigma_0^{(i)}\gamma_0^{(i)}) + \Big(-\frac{3}{4}\sigma_k^{(i)2} - \frac{3}{4} + \frac{1}{2}s_{i,3}^2\Big)\sin 2(E_k-E_0)\Big], \end{aligned} \tag{1}$$

$$\sigma_k^{(i)} = s_{i,1}\sin E_k - s_{i,2}\cos E_k,$$

$$\gamma_k^{(i)} = s_{i,1}\cos E_k + s_{i,2}\sin E_k,$$

$s_{i,1}$, $s_{i,2}$, $s_{i,3}$ are the planned coefficients (in the generally accepted denotations P, Q, R), n is the daily motion of the body, h is the integration step, E_k is the eccentric anomaly (in this case the mean anomaly, since e = 0), ρ is the maximum rounding off error at computations of the right-hand parts of equations in each step.

Such is the form in which the formulas are utilized by us for the quantitative comparison of the error $\delta_k^{(i)}$, obtained at numerical integration with a forecast $\varepsilon_k^{(i)}$.

1. - Let us perform the indicated comparison on examples, especially computed to that effect (the examples were computed with aid of the computer of the BESM Academy of Sciences of the USSR). The problem resolved is that of a plane unperturbed motion with different initial data (orbit elements).

The value of the integration interval and the elements are so chosen that exactly one hundred steps are made over one convolution. Three examples are computed in all with 1100 steps in each of them; the initial conditions are determined by the following elements :

	Example I	Example II	Example III
M_0	0°	0°	0°
ω	0°	0°	0°
e	0.04825380	0.04825380	0.2
n	$299''.128376$	$648''$	$648''$
Integr. step h ...	$43^d.35258794$	$20^d.0$	$20^d.0$

The integration was performed by the Cowell quadrature method taking into account the fourth differences

$$\bar{X} = \bar{f}^{(-2)} + \frac{1}{12}\bar{f} - \frac{1}{240}\bar{f}^{(2)} + \frac{31}{60480}\bar{f}^{(4)},$$

where

$$\bar{f} = -h^2k^2\frac{\bar{X}}{r^3},$$

and $\bar{X}$ is a vector with components X, Y, Z.

The results of integration

$$\bar{X}_1,\ \bar{X}_2,\ \ldots,\ \bar{X}_k,\ \ldots$$

were compared with the quantities

$$\bar{x}_1,\ \bar{x}_2,\ \ldots,\ \bar{x}_k,\ \ldots,$$

earlier computed by the elliptic motion formulas in the sixth decimal point the difference

$$\bar{x}_k - \bar{X}_k = \bar{\delta}_k$$

was taken for the pure accumulation of rounding off errors, inasmuch as the influence of higher differences in the integration formula, in the given case $f^{(6)}$, is taken into account with the maximum precision

$$\left(\frac{317}{22809600}f^{(6)} < 1 \cdot 10^{-11}\right).$$

In the following we shall neglect these errors too, for their influence in all the considered works lies beyond the limits of the precision with which the computations are performed.

The quantity $0.5 \cdot 10^{-9}$ was taken for the rounding off error $\underline{\rho}$ that is, the precision with which $\underline{f}$ were computed in each step.

It may be noted that formulas (1) will be significantly simplified if we compute the estimates for the points $E_k - E_0$, multiples of $0, \frac{\pi}{2}, \pi, \frac{3}{2}\pi$. As to our case, formulas (1) will be still further simplified because the problem resolved was that of plane motion ($s_{11} = s_{22} = 1, s_{12} = s_{21} = 0$) and the integration began from the perihelion point ($E_0 = 0$); namely

$$|\delta_k^{(i)}| < \varepsilon_k^{(i)} = \frac{\sqrt{3}}{(nh)^{\frac{3}{2}}} \rho \sqrt{N^{(i)}(E_k)} \qquad (i = 1,\ 2;\ k = 1,\ 2,\ 3 \ldots),$$

$$N^{(1)}(E_k) = \frac{13}{2} E_k, \quad N^{(2)}(E_k) = 3E_k^3 + 38E_k \qquad \text{at } E_k = 2m\pi,\ (m = 1, 2 \ldots)$$

$$N^{(1)}(E_k) = 3E_k^3 + 14E_k - 32, \quad N^{(2)}(E_k) = \frac{13}{2} E_k - 8 \quad \text{at } E_k = \left(2m + \frac{1}{2}\right)\pi, \tag{2}$$

$$N^{(1)}(E_k) = \frac{13}{2} E_k, \quad N^{(2)}(E_k) = 3E_k^3 - 10E_k \qquad \text{at } E_k = (2m + 1)\pi,$$

$$N^{(1)}(E_k) = 3E_k^3 + 14E_k + 32, \quad N^{(2)}(E_k) = \frac{13}{2} E_k + 8 \quad \text{at } E_k = \left(2m + \frac{3}{2}\right)\pi.$$

Beginning with $k > 200$, the ratio $\frac{E_k}{E_k^3}$ becomes of the order $1/_{100}$ and that is why, neglecting the first power E_k by comparison with E_k^3, and assuming $E_k = nhk$, we shall obtain

$$\begin{aligned} &\varepsilon_k^{(1)} = 70.3\rho k^{\frac{1}{2}}, \quad \varepsilon_k^{(2)} = 3\rho k^{\frac{3}{2}} \qquad &&\text{at } E_k = m\pi, \\ &\varepsilon_k^{(1)} = 3\rho k^{\frac{3}{2}}, \quad \varepsilon_k^{(2)} = 70.3\rho k^{\frac{1}{2}} \qquad &&\text{at } E_k = \left(m + \frac{1}{2}\right)\pi. \end{aligned} \tag{3}$$

Note that $\varepsilon_k^{(i)}$ are obtained identical for all the three considered examples. This is due to the fact that the product <u>nh</u> is the same $\left(\frac{\pi}{50}\right)$ in all examples, and that the influence of eccentricity does not distort the result very much. At $k > 200$ the formula for the estimate (Myachin) is written in the form

$$\varepsilon_k = 3\rho\, |\sigma_k|\, \frac{k^{\frac{3}{2}}}{1 - e \cos E_k}.$$

The computed $|\delta_k^{(i)}|$ and the forecast errors $\varepsilon_k^{(i)}$ are compiled in Table 1. We placed in the first column the mean anomaly for all the three "planets", in the second – the respective number of integration steps; the true errors $\delta_k^{(1)}$ and $\delta_k^{(2)}$ for the examples I–III are in columns 3 – 8, while the 9th and the 10th contain the values of $\varepsilon_k^{(1)}$ and $\varepsilon_k^{(2)}$, providing the error estimate in all the three examples. The values of $\delta_k^{(i)}, \varepsilon_k^{(i)}$ are expressed in units of the sixth decimal sign.

T A B L E 1

Anoma-lies E_k	Number of steps	Example I $\delta_k^{(1)}$	Example I $\delta_k^{(2)}$	Example II $\delta_k^{(1)}$	Example II $\delta_k^{(2)}$	Example III $\delta_k^{(1)}$	Example III $\delta_k^{(2)}$	Estimate $\varepsilon_k^{(1)}$	Estimate $\varepsilon_k^{(2)}$
0	1	0	0	0	0	0	0	0	0
90	26	0	0	0	0	0	0	0	0
180	51	1	1	0	0	0	0	0	0
270	76	3	0	0	0	0	0	0	0
360	101	0	5	0	0	0	+ 1	0	1.5
450	126	3	2	0	0	0	0	2	0
540	151	0	3	0	1	1	3	0	3
630	176	6	0	0	0	0	0	4	0
720	201	0	7	0	0	0	1	0	4
810	226	6	0	+ 1	0	1	0	5	0
900	251	0	5	0	0	0	5	0	6
990	276	7	0	0	0	0	0	7	0
1080	301	0	9	0	0	0	0	1	8
1170	326	7	0	0	− 1	+ 3	0	9	1
1260	351	0	5	0	− 2	0	− 6	1	10
1350	376	− 7	− 1	1	− 1	− 2	− 3	11	1
1440	401	0	−11	0	− 1	0	− 1	1	12
1530	426	8	− 1	− 1	− 1	− 1	2	13	2
1620	451	1	5	0	− 2	0	4	1	14
1710	476	− 8	− 1	1	− 1	− 2	− 3	15	2
1800	501	0	−10	0	− 1	0	− 8	1	16
1890	526	6	− 2	− 1	− 1	3	1	18	2
1980	551	0	3	0	− 2	0	− 2	2	19
2070	576	− 6	− 1	0	− 1	− 7	− 1	21	0
2160	601	0	7	0	− 1	0	−14	2	22
2250	626	1	− 2	− 1	1	7	1	24	2
2340	651	− 1	− 1	0	− 1	0	− 2	2	25
2340	676	− 1	− 1	− 1	+ 1	−10	− 1	26	2
2520	701	0	− 3	0	− 3	0	−21	2	27
2610	726	− 2	− 3	2	− 1	12	− 1	28	2
2700	751	2	− 7	0	1	0	5	2	30
2790	776	4	− 2	− 4	− 1	−15	− 1	31	2
2880	801	0	2	0	− 7	0	−29	2	33
2970	826	− 6	− 4	6	− 1	16	− 1	35	2
3060	851	2	−11	0	4	0	8	3	36
3150	876	9	− 2	− 8	− 1	−20	0	38	3
3240	901	0	7	0	−12	0	−35	3	40
3330	926	−11	− 3	9	− 1	20	− 1	42	3
3420	951	− 2	−16	0	9	0	12	3	44
3510	976	14	0	−13	− 1	−25	1	46	3
3600	1001	0	12	0	−19	0	−44	3	48
3690	1226	−15	− 4	18	− 1	36	− 1	50	3
3780	1051	− 2	−21	0	15	0	16	3	52
3870	1076	19	− 2	−20	0	−31	2	53	3
3960	1101	0	17	0	−26	0	−53	3	54

Comparing the 3rd, 5th and 7th columns with the 9th, or the 4th, 6th, 8th with 10th, we shall be able to judge on the quality of the obtained estimate. Attention should be called to the fact that the estimate reflects the fluctuating character of error accumulation.

2.- From the standpoint of error accumulation it was found to be interesting to consider the coordinate of Uranus, Saturn, Jupiter, obtained by D.K.Kulikov when integrating VIII Jupiter satellite for the period from 24 January 1930 to 28 August 1965. The integration was performed on the BESM computer with a 10-day step (altogether 1300 steps). The planets' coordinates were obtained at simultaneous integration of a system of nine equations, the initial coordinates being borrowed from the Astronomical Papers of 1951.

Inasmuch as the coordinates of the planets, published in 1951 in "Astronomical Papers", were computed with great precision (a system of equations of motion were jointly resolved for 5 outer planets, the calculations having been performed with 14 columns), they were accepted as a precise solution of $\bar{x}_k$, and the difference $\bar{x}_k - \overline{X}_k$ was taken for the true error in the estimated coordinates. The comparison was made in the fifth decimal point. The results are compiled in Table 2.

For the estimate of this error by V.F.Myachin formulas it is necessary to establish the error with which the calculations are performed on one step. In the given case, aside from the rounding off error $\rho (\rho = 0.5 \cdot 10^{-9})$, there will be at computations of right-hand parts of the equations an error on account of disregarded perturbations of Neptune and Pluto. The values of perturbations reach $1 \cdot 10^{-9}$, that is, they exceed the calculation error. Of the three planets Uranus is the one subject to greatest influence of perturbations, for during the investigated time interval Uranus has time to effect only half of a convolution, and the aggregate perturbation from Neptune and Pluto has a constant sign in the course of a significant time period while Saturn effects 4.5 convolution and Jupiter — 3.5. The perturbations from Neptune and Pluto have a periodic character. For the estimate of error on account of rounding off we shall make use of formulas (1) (the results are shown in columns 4, 7 and 10 of Table 2). These formulas

TABLE 2

Anomalies E_k	Number of steps	$x_k - X_k$	Round-off errors	Erros for unaccounted perturbations	$y_k - Y_k$	Round-off errors	Errors for unaccounted perturbations	$z_k - Z_k$	Round-off errors	Errors for unaccounted perturbations
					JUPITER					
135	90	0	0.1	1	0	0.1		0	0.0	
180	144	0	0.2	1	0	0.3	5	0	0.1	2
225	198	−2	0.4		0	0.1		0	0.8	
270	252	−3	0.5	15	− 1	0.3	3	0	0.1	1
315	306	−2	0.3		− 3	0.8		−1	0.3	
360	360	+1	0.5	7	− 3	1.1	31	−1	0.5	16
405	414	+2	1.3		− 2	0.6		0	0.3	
450	468	+3	1.6	56	0	0.6	11	0	0.5	4
495	522	+1	0.9		2	1.5		1	0.6	
540	576	−2	0.7	16	2	1.8	77	0	0.8	53
585	630	−3	2.1		2	1.0		0	0.5	
630	684	−6	2.6	120	− 1	0.8	24	0	0.3	8
675	738	−4	1.4		− 6	2.5		−2	1.1	
720	792	0	1.1	32	− 8	3.0	147	−4	1.3	63
765	846	5	3.4		− 4	1.6		−2	0.8	
810	900	5	4.0	207	0	1.2	42	0	0.4	13
855	954	2	2.2		4	3.8		2	1.5	
900	1008	−1	1.4	51	4	4.3	239	2	1.9	103
945	1066	−5	4.6		2	0.7		1	0.2	
990	1120	−8	5.4	319	− 2	1.5	65	−1	0.5	20
1035	1174	−5	3.0		−10	5.0		−4	2.0	
1080	1228	+4	4.5	76	−10	5.0	353	−4	2.5	152
1125	1272	+8	6.1		− 6	3.6		−3	2.2	
					URANUS					
225	120	+ 1	0.2		0	0.4		0	0.2	
270	256	+ 3	0.6	0.6	− 1	1.0	16	0	0.4	6
315	392	+ 9	1.4		− 1	1.1		−1	0.5	
360	528	+19	1.4	72	+ 6	0.9	4	+1	0.4	1.4
405	664	+16	1.2		+20	2.4		+7	1.0	
450	800	+ 1	1.2	6.4	+22	3.4	154	+9	1.4	64
495	936	− 9	4.0		+12	2.6		+5	1.2	
540	1072	−17	6.2	299	− 9	1.4	15	−4	0.6	6
					PLUTO					
225	213	+ 2	3.3	7	− 1	4.1	8	− 1	2.0	4
270	587	+14	2.5	60	− 7	1.5	12	− 3	1.0	7
315	970	+31	5.2	188	−23	4.8	150	−11	4.8	47

cannot be used for the evaluation of error due to unaccounted perturbations, for the latter are subject to the law of random errors, while in deriving (1) the probable law of random error distribution was utilized. The only acceptable formula in this case may be the following (Myachin, 1959):

$$|\delta_k^{(i)}| < \varepsilon_k^{(i)} \approx \frac{3}{2}\sqrt{3}\,\rho\,|\sigma_k^{(i)}|\,k^2 \qquad (i=1,\ 2,\ 3;\ k=1,\ 2,\ 3\ldots), \tag{4}$$

where we took for ρ the error on one step on account of disregarded perturbations, and $\sigma_k^{(i)}$ has the same value as in (1).

When computing by formulas (4) we took for ρ the maximum perturbation from Neptune and Pluto for all the three planets, that is, $1 \cdot 10^{-9}$, which obviously gives a strong overrating for Jupiter and Saturn (Table 2, columns 5, 8, 11).

3.- We shall give one more example. We will attempt to estimate the error resulting from round-off errors in the coordinates of major planets published in the "Astronomical Papers" of 1951. The estimate will be made according to the rough formula obtained from (1) at the following assumptions: we neglect E_k^2 and E_k by comparison with E_k^3, and we assume σ_k^i equal to 1. Then (1) will take the form

$$|\delta_k^i| < \varepsilon_k^i \approx 3\rho k^{\frac{3}{2}}. \tag{5}$$

If we take for ρ $1 \cdot 10^{-14}$, then, after 100 integration steps, which correspond in the given case to more than 100 years, the error in planet coordinates will be about $1 \cdot 10^{-9}$, that is, the published coordinates of major planets are free from round-off errors.

Therefore, the above-presented examples show that the formulas derived by V. F. Myachin for the estimate of round-off errors are quite valid for practical utilization.

The estimate (1) reflects the fluctuating character of the error and gives a comparatively small overrating (as a rule, less than 10 times). It is then revealed that after 1000 integration steps no more than 5 columns are lost in the quantity searched for on account of round-of errors.

As to the error caused by unaccounted perturbations (Table 2), the estimate (4) utilized for them should be considered as unsatisfactory, inasmuch as it does not take into account the sign-changing character of perturbations. This estimate gives a practically acceptable result only in the case when the perturbations indicate, over the entire integration interval or over its greater portion, values of constant sign.

*** THE END ***

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